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RN-003-1015001

Seat No. ______

B. Sc. (Sem. V) (CBCS) Examination

February - 2019

Mathematics : Paper - 5(A)

(Mathematical Analysis - I & Abstract Algebra - I)

(New Course)

Faculty Code : 003

Subject Code : 1015001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) All questions are compulsory.
(2) Figure to the right indicate full marks of the question.

1 (a) Answers the following questions : **4**

(1) Define : Metric space.

(2) If $(\mathbb{R}, d)$ is a discrete metric space, then find $N(e, \pi)$.

(3) If $(\mathbb{R}, d)$ is a usual metric space, then find $\mathbb{Z}'$.

(4) Let $A = \left\{\frac{1}{2} \middle| n \in \mathbb{N}\right\} \subset \mathbb{R}$. Find $\overline{A}$.

(b) Answer any **one** in brief : **2**

(1) By an example show that : arbitrary union of closed sets may not be closed.

(2) By an example show that : arbitrary intersection of open sets may not be open.

(c) Answer any **one** in detail : **3**

(1) Prove that if (X, d) is a metric space, then $|d(x, z) - d(y, z)| \le d(x, y), \forall x, y, z \in X$

(2) Prove that : Any finite subset of a metric space is closed.

(d) Attempt any **one** : **5**

(1) If (X, d) is a metric space, then show that $d_1 : X \times X \to \mathbb{R}; d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ is a bounded metric on X.

(2) Lex X be a metric space and $A, B \subset X$. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2 (a) Answers the following questions : 4

(1) In usual notation define $L(P, f)$.

(2) Let $f(x) = x, x \in [0, 1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of [0, 1]. Compute $L(P, f)$.

(3) True or False : If a bounded function f is monotonic, then it is R-integrable.

(4) State First Mean Value Theorem of Integral Calculus.

(b) Answer any **one** in brief : 2

(1) If $f : [a, b] \to \mathbb{R}$ is a bounded and $P \in P[a, b]$, then show that $L(P, f) \le U(P, f)$.

(2) If $f : [a, b] \to \mathbb{R}$ is a bounded and $P \in P[a, b]$, then show that $L(P, -f) = -U(P, f)$.

(c) Answer any **one** in detail : 3

(1) Show that : $\lim\limits_{x \to \infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ... + \frac{1}{8n}\right] = \frac{3}{8}$

(2) Using second definition prove that $\int_1^2 (2x+1)\,dx = 4$.

(d) Attempt any **one** : 5

(1) Prove that : $\frac{1}{3\sqrt{2}} \le \int_0^1 \frac{x^2}{\sqrt{1+x^2}} \le \frac{1}{3}$.

(2) Let $f : [0, 1] \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} \frac{1}{2} & x \text{ is rational} \\ \frac{1}{3} & x \text{ is irrational} \end{cases}$$

Show that f is not R-integrable.

3 (a) Answers the following questions : 4

(1) In usual notation define $U(P, f)$.

(2) Define : Norm of the partition.

(3) Let * be defined as $a * b = \frac{ab}{100}, a, b \in \mathbb{Q}$. What is the identity for * ?

(4) Let G be a group and $x, y, z, w \in G$. Then $(x\,y\,z\,w)^{-1} =$ __________.

(b) Answer any **one** in brief : **2**

(1) Express $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{3r}{n} - 2 \right)$ as a definite integral.

(2) In a group G prove that inverse of every element is unique.

(c) Answer any **one** in detail : **3**

(1) Let G be a group $H \leq G$ and $a \in G$. Show that aHa^{-1} is a subgroup of G.

(2) If $f \in R[a, b]$ and $f(x) \geq 0 \ \forall x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.

(d) Attempt any **one** : **5**

(1) Show that :

$$\lim_{n\to\infty} \frac{1}{n}\left(e^{3/n} + e^{6/n} + \ldots.. + e^{3n/n}\right) = \frac{1}{3}\left(e^3 - 1\right)$$

(2) Let G be a group and $H \leq G$. For $x, y \in G$ define $x \equiv y \pmod H \Leftrightarrow x^{-1}y \in H$. Prove that $\equiv$ is an equivalence relation on G. Also find *cl(a)* where $a \in G$.

4 (a) Answers the following questions : **4**

(1) Define : Group.

(2) Define : Order of an element.

(3) Express $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 1 & 5 & 4 & 2 & 7 \end{pmatrix} \in S_7$ as a product of disjoint cycles.

(4) Let G be a group with identity e and $a \in G$. If $a^9 = e$, them list all possible orders of element a.

(b) Answer any **one** in brief : **2**

(1) Let G be a group. If $a^2 = e, \forall a \in G$, then show that G is abelian.

(2) Let G be a finite group and $a \in G$. Then show that $a^{O(G)} = e$.

(c) Answer any **one** in detail : **3**

(1) Using Fermat's Theorem Show that : If p is an odd prime, then

$$1^{p-1} + 2^{p-1} + \ldots. + (p-1)^{p-1} \equiv (-1) \pmod p$$

(2) Let G be a group. Show that if $(xy)^{-1} = x^{-1}y^{-1}, \forall x, y \in G$, then G is abelian.

(d) Attempt any **one** : **5**

(1) Prove that : For $n \geq 3$, every $f \in A_n$ can be express as product of 3-cycle.

(2) State and prove Euler's Theorem.

5 (a) Answers the following questions : **4**

(1) Define : Transposition.

(2) Define : Center of the group.

(3) Check whether $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 1 & 4 & 6 & 5 & 2 \end{pmatrix} \in S_7$ is odd or even ?

(4) Let G be a group, $a, b \in G$ and $O(a) = 6$. If $c = bab^{-1}$, then $O(c)$ = ________.

(b) Answer any **one** in brief : **2**

(1) Let G be an abelian group. Show that $H = \{a \in G \mid O(a) \text{ is finite}\}$ is a subgroup of G.

(2) Let G be a group and $a \in G$. Then show that $N(a) \leq G$.

(c) Answer any **one** in detail : **3**

(1) Find the remainder obtained on dividing 5^{352} by 14.

(2) Prove that product of an even and an odd permutation is an odd permutation.

(d) Attempt any **one** : **5**

(1) State and prove Cayley's Theorem.

(2) If G is a group such that $(ab)^n = a^n b^n$, for three consecutive integers n, then show that G is abelian.